INFINITE VOLUME AND ATOMS AT THE BOTTOM OF THE SPECTRUM

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ABSTRACT. Let G be a higher rank simple real algebraic group, or more generally, any semisimple real algebraic group with no rank one factors and X the associated Riemannian symmetric space. For any Zariski dense discrete subgroup $\Gamma < G$, we prove that $\mathrm{Vol}(\Gamma\backslash X) = \infty$ if and only if no positive Laplace eigenfunction belongs to $L^2(\Gamma\backslash X)$, or equivalently, the bottom of the L^2-spectrum is not an atom of the spectral measure of the negative Laplacian.

1. INTRODUCTION

Let $\mathcal{M}$ be a complete Riemannain manifold and let Δ denote the Laplace-Beltrami operator on $\mathcal{M}$. Define the real number $\lambda_0(\mathcal{M}) \in [0, \infty)$ by

$$\lambda_0(\mathcal{M}) := \inf \left\{ \frac{\int_{\mathcal{M}} \|\mathrm{grad}\, f\|^2 \, d\,\mathrm{vol}}{\int_{\mathcal{M}} |f|^2 \, d\,\mathrm{vol}} \; : \; f \in C_c^\infty(\mathcal{M}) \right\}, \tag{1.1}$$

where $C_c^\infty(\mathcal{M})$ denotes the space of all smooth functions with compact supports. This number $\lambda_0(\mathcal{M})$ is known as the bottom of the L^2-spectrum of the negative Laplacian $-\Delta$ and separates the L^2-spectrum and the positive spectrum [24, p. 329] (Fig. 1). More precisely, let $L^2(\mathcal{M})$ denote

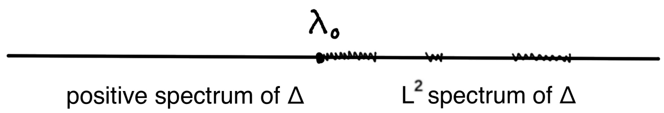

FIGURE 1. λ_0 separates the L^2 and positive spectrum

the space of all square-integrable functions with respect to the inner product $\langle f_1, f_2 \rangle = \int_{\mathcal{M}} f_1 f_2 \, d\,\mathrm{vol}$. Let $W^1(\mathcal{M}) \subset L^2(\mathcal{M})$ denote the closure of $C_c^\infty(\mathcal{M})$ with respect to the norm

$$\|f\|_{W^1} = \left(\int_{\mathcal{M}} f^2 \, d\,\mathrm{vol} + \int_{\mathcal{M}} \|\,\mathrm{grad}\, f\|^2 \, d\,\mathrm{vol}\right)^{1/2}.$$

There exists a unique self-adjoint operator on the space $W^1(\mathcal{M})$ extending the Laplacian Δ on $C_c^\infty(\mathcal{M})$, which we also denote by Δ (cf. [12, Chapter

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4.2]). The L^2-spectrum of $-\Delta$ is the set of all $\lambda \in \mathbb{C}$ such that $\Delta + \lambda$ does not have a bounded inverse $(\Delta + \lambda)^{-1} : L^2(\mathcal{M}) \to W^1(\mathcal{M})$. Sullivan showed that the L^2-spectrum of $-\Delta$ contains $\lambda_0(\mathcal{M})$ and is contained in the positive ray $[\lambda_0(\mathcal{M}), \infty)$, that is, $\lambda_0(\mathcal{M})$ is the bottom of the L^2-spectrum, and moreover, there are no positive eigenfunctions with eigenvalue strictly bigger than $\lambda_0(\mathcal{M})$ [24, Theorem 2.1 and 2.2] (see Fig. 1). We will call an eigenfunction with eigenvalue $\lambda_0(\mathcal{M})$ a *base eigenfunction*. Note that the absence of a base eigenfunction in $L^2(\mathcal{M})$ is same as the absence of a positive eigenfucntion in $L^2(\mathcal{M})$.

In this paper, we are concerned with locally symmetric spaces. Let G be a connected semisimple real algebraic group and (X, d) the associated Riemannian symmetric space. Let $\Gamma < G$ be a discrete torsion-free subgroup and let $\mathcal{M} = \Gamma\backslash X$ the corresponding locally symmetric manifold.

For a rank one locally symmetric manifold $\mathcal{M} = \Gamma\backslash X$, the relation between $\lambda_0(\mathcal{M})$ and the critical exponent[1] δ_Γ is well-known: if we denote by $D = D_X$ the volume entropy of X, then

$$\lambda_0(\mathcal{M}) = \begin{cases} D^2/4 & \text{if } \delta_\Gamma \leq D/2 \\ \delta_\Gamma(D - \delta_\Gamma) & \text{otherwise} \end{cases}$$

([7]-[9], [18]-[20], [24], [4]). We refer to ([16], [1], [3]) for extensions of these results in higher ranks. We remark that when G has Kazhdan's property (T) (cf. [28, Theorem 7.4.2]), we have $\mathrm{Vol}(\mathcal{M}) = \infty$ if and only if $\lambda_0(\mathcal{M}) > 0$ ([4], [16]).

The goal of this article is to study the square-integrability of a base eigenfunction of locally symmetric manifolds. The space of square-integrable base eigenfunctions is at most one dimensional and generated by a *positive* function when non-trivial [24]. Based on this positivity property and using their theory of conformal measures on the geometric boundary, Patterson and Sullivan showed that if $\mathcal{M}$ is a geometrically finite real hyperbolic $(n+1)$-manifold, then $\mathcal{M}$ has a square-integrable base eigenfunction if and only if the critical exponent δ_Γ is strictly greater than $n/2$ ([21], [25], [24, Theorem 2.21]). More generally, the formula for $\lambda_0(\mathcal{M})$ given above, together with [13, Corollary 3.2] (cf. also [17]) and [27, Theorem 1.1], imply that any rank one geometrically finite manifold $\mathcal{M}$ has a square-integrable base eigenfunction if and only if the critical exponent δ_Γ is strictly greater than $D_X/2$.

The main theorem of this paper is the following surprising higher rank phenomenon that contrasts with the rank one situation:

Theorem 1.1. *Let G be a connected semisimple real algebraic group with no rank one factors. For any Zariski dense discrete torsion-free subgroup $\Gamma < G$, we have* $\mathrm{Vol}(\Gamma\backslash X) = \infty$ *if and only if* $\Gamma\backslash X$ *does not possess any*

[1] the abscissa of convergence of the Poincare series $s \mapsto \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)}$, $o \in X$.

square-integrable positive Laplace eigenfunction, that is, $\lambda_0(\Gamma\backslash X) > 0$ *is not an atom for the spectral measure of* $-\Delta$*.*

In other words, when $\mathrm{Vol}(\Gamma\backslash X) = \infty$, no base eigenfunction is square-integrable (see also Theorem 4.3 for a more general version). A special case of this theorem for Anosov subgroups of higher rank semisimple Lie groups was proved in [10, Theorem 1.8]. See Theorem 4.3 for a more general version.

Our proof of Theorem 1.1 is based on the higher rank version of Patterson-Sullivan theory introduced by Quint [22], with a main new input being the recent theorem of Fraczyk and Lee (Theorem 4.1, [11]). Suppose that $\mathrm{Vol}(\Gamma\backslash X) = \infty$ and a base eigenfunction is square-integrable. Using Sullivan's work [24], it was then shown by Edwards and Oh [10] that there exists a Γ-conformal density $\{\nu_x : x \in X\}$ on the Furstenberg boundary of G (see Definition 2.1) such that any such base eigenfunction is proportional to the function E_ν given by

$$E_\nu(x) = |\nu_x| \quad \text{for all } x \in X. \tag{1.2}$$

Moreover, the following higher rank version of the smearing theorem of Thurston and Sullivan ([25], [26]) was also obtained in [10] (see Theorem 3.1):

$$|\mathsf{m}_{\nu,\nu}| \ll \int_{\Gamma\backslash X} |E_\nu|^2 dx,$$

where $\mathsf{m}_{\nu,\nu}$ is a generalized Bowen-Margulis-Sullivan measure on $\Gamma\backslash G$ corresponding to the pair (ν, ν); see Definition 3.3. On the other hand, the recent theorem of Fraczyk and Lee (Theorem 4.1, [11]) which describes all discrete subgroups admitting finite BMS measures implies that $|\mathsf{m}_{\nu,\nu}| = \infty$, and consequently, $E_\nu \notin L^2(\Gamma\backslash X)$, yielding a contradiction. We remark that the integrand on the right hand side of (1.2) can be replaced by an $O(1)$-neighborhood of the support of $\mathsf{m}_{\nu,\nu}$ and Sullivan used the rank one version of this to deduce the finiteness of the BMS measure $\mathsf{m}_{\nu,\nu}$ attached to the (unique) Patterson-Sullivan measure ν from the the growth control of the base eigenfunction for Γ geometrically finite [25].

We close the introduction by presenting two related questions on the L^2-spectrum. When $\Gamma < G$ is geometrically finite in a rank one Lie group and there is no positive square-integrable eigenfunction, there are no Laplace eigenfunctions in $L^2(\Gamma\backslash X)$ and the quasi-regular representation $L^2(\Gamma\backslash G)$ is tempered[2] ([18], [25], [5], [15]). In view of this, we ask the following question: let G be a semisimple real algebraic group with no rank one factors and $\Gamma < G$ be a Zariski dense discrete subgroup.

Question 1.1. (1) When $\Gamma < G$ is not a lattice, can there exist any Laplace eigenfunction in $L^2(\Gamma\backslash X)$?
(2) Is there an example of Γ such that $L^2(\Gamma\backslash G)$ is non-tempered?

[2]This means that $L^2(\Gamma\backslash G)$ is weakly contained in $L^2(G)$, or equivalently, every matrix coefficient of $L^2(\Gamma\backslash G)$ is $L^{2+\varepsilon}(G)$-integrable for any $\varepsilon > 0$.

Regarding the question (2), there are many non-Zariski dense discrete subgroups such that $L^2(\Gamma\backslash G)$ is non-tempered. For example, if $H < G$ is a connected semisimple subgroup such that $L^2(H\backslash G)$ is not tempered (e.g. $G = \mathrm{SL}_{2n}(\mathbb{R})$ and $H = \mathrm{Sp}_{2n}(\mathbb{R})$, $n \geq 2$, satisfy this and see [2, Section 5] for more examples of such H and G) and $\Gamma < H$ is a lattice in H, then $L^2(\Gamma\backslash G)$ is non-tempered. On the other hand, for Zariski dense Hitchin subgroups $\Gamma < \mathrm{PSL}_n(\mathbb{R})$, $L^2(\Gamma\backslash \mathrm{PSL}_n(\mathbb{R}))$, $n \geq 3$, is tempered [10, Theorem 1.7].

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2. Positive eigenfuntions and conformal measures

Let G be a connected semisimple real algebraic group. We fix, once and for all, a Cartan involution θ of the Lie algebra $\mathfrak{g}$ of G, and decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of θ, respectively. We denote by K the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. We also choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. Fixing a left G-invariant and right K-invariant Riemannian metric on G induces a Weyl-group invariant inner product and corresponding norm on $\mathfrak{a}$, which we denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. Note also that the choice of this Riemannian metric induces a G-invariant metric $d(\cdot, \cdot)$ on G/K. We denote by $X = G/K$ the corresponding Riemannian symmetric space. The Riemannian volume form on X is denoted by $d\,\mathrm{vol}$. We also use dx to denote this volume form, as well as for the Haar measure on G.

Let $A := \exp \mathfrak{a}$. Choosing a closed positive Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$, let $A^+ = \exp \mathfrak{a}^+$. The centralizer of A in K is denoted by M, and we set N to be the maximal horospherical subgroup for A so that $\log(N)$ is the sum of all positive root subspaces for our choice of $\mathfrak{a}^+$. We set $P = MAN$, which is a minimal parabolic subgroup of G. The quotient

$$\mathcal{F} = G/P$$

is known as the Furstenberg boundary of G, and since K acts transitively on $\mathcal{F}$ and $K \cap P = M$, we may identify $\mathcal{F}$ with K/M.

Let Σ^+ denote the set of all positive roots for $(\mathfrak{g}, \mathfrak{a}^+)$. We also write $\Pi \subset \Sigma^+$ for the set of all simple roots. For any $g \in G$, there exists a unique element $\mu(g) \in \mathfrak{a}^+$ such that $g \in K \exp \mu(g) K$. The map $\mu : G \to \mathfrak{a}^+$ is called the Cartan projection. Setting $o = [K] \in X$, we then have $\|\mu(g)\| = d(go, o)$ for all $g \in G$. Throughout the paper we will identify functions on X with right K-invariant functions on G. For each $g \in G$, we define the following *visual* maps:

$$g^+ := gP \in \mathcal{F} \quad \text{and} \quad g^- := gw_0P \in \mathcal{F}, \tag{2.1}$$

where w_0 denotes the Weyl group element such that $\mathrm{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$. The unique open G-orbit $\mathcal{F}^{(2)}$ in $\mathcal{F} \times \mathcal{F}$ under the diagonal G-action is given by $\mathcal{F}^{(2)} = G(e^+, e^-) = \{(g^+, g^-) \in \mathcal{F} \times \mathcal{F} : g \in G\}$. Let $G = KAN$ be the

Iwasawa decomposition, and define the Iwasawa cocycle $H : G \to \mathfrak{a}$ by the relation:

$$g \in K \exp\big(H(g)\big)N.$$

The $\mathfrak{a}$-valued Busemann map is defined using the Iwasawa cocycle as follows: for all $g \in G$ and $[k] \in \mathcal{F}$ with $k \in K$, define

$$\beta_{[k]}(g(o), h(o)) := H(g^{-1}k) - H(h^{-1}k) \in \mathfrak{a} \quad \text{for all } g, h \in G.$$

Conformal measures. We denote by $\mathfrak{a}^*$ the space of all real-valued linear forms on $\mathfrak{a}$. In the rest of this section, let $\Gamma < G$ be a discrete subgroup. The following notion of conformal densities was introduced by Quint [22, Section 1.2], generalizing Patterson-Sullivan densities for rank one groups ([21, Section 3], [23, Section 1]).

Definition 2.1. Let $\psi \in \mathfrak{a}^*$.

(1) A finite Borel measure ν on $\mathcal{F} = K/M$ is said to be a (Γ, ψ)-conformal measure (for the basepoint o) if for all $\gamma \in \Gamma$ and $\xi = [k] \in K/M$,

$$\frac{d\gamma_* \nu}{d\nu}(\xi) = e^{-\psi(\beta_\xi(\gamma o, o))},$$

where $\gamma_*\nu(Q) = \nu(\gamma^{-1}Q)$ for any Borel subset $Q \subset \mathcal{F}$.

(2) A collection $\{\nu_x : x \in X\}$ of finite Borel measures on $\mathcal{F}$ is called a (Γ, ψ)-conformal density if, for all $x, y \in X$, $\xi \in \mathcal{F}$ and $\gamma \in \Gamma$,

$$\frac{d\nu_x}{d\nu_y}(\xi) = e^{-\psi(\beta_\xi(x,y))} \quad \text{and} \quad d\gamma_*\nu_x = d\nu_{\gamma(x)}. \tag{2.2}$$

A (Γ, ψ)-conformal measure ν defines a (Γ, ψ)-conformal density $\{\nu_x : x \in X\}$ by the formula:

$$d\nu_x(\xi) = e^{-\psi(\beta_\xi(x,o))} d\nu(\xi),$$

and conversely any (Γ, ψ)-conformal density $\{\nu_x\}$ is uniquely determined by its member ν_o by (2.2). By a Γ-conformal measure on $\mathcal{F}$, we mean a (Γ, ψ)-conformal measure for some $\psi \in \mathfrak{a}^*$.

Definition 2.2. Let $\psi \in \mathfrak{a}^*$. Associated to a (Γ, ψ)-conformal measure ν on $\mathcal{F}$, we define the following function E_ν on G: for $g \in G$,

$$E_\nu(g) := |\nu_{g(o)}| = \int_{\mathcal{F}} e^{-\psi\left(H(g^{-1}k)\right)}\, d\nu([k]). \tag{2.3}$$

Since $|\nu_{\gamma(x)}| = |\nu_x|$ for all $\gamma \in \Gamma$ and $x \in X$, the left Γ-invariance and right K-invariance of E_ν are clear. Hence we may consider E_ν as a K-invariant function on $\Gamma \backslash G$, or, equivalently, as a function on $\Gamma \backslash X$.

Let $\mathcal{D} = \mathcal{D}(X)$ denote the ring of all G-invariant differential operators on X. For each (Γ, ψ)-conformal measure ν, E_ν is a joint eigenfunction of $\mathcal{D}$ and conversely, any *positive* joint eigenfunction on $\Gamma \backslash X$ arises as E_ν for some (Γ, ψ)-conformal measure ν [10, Proposition 3.3].

Let Δ denote the Laplace-Beltrami operator on X or on $\Gamma\backslash X$. Since Δ is an elliptic differential operator, an eigenfunction is always smooth. We say a smooth function f is λ-harmonic if

$$-\Delta f = \lambda f.$$

Define the real number $\lambda_0 = \lambda_0(\Gamma\backslash X) \in [0, \infty)$ as follows:

$$\lambda_0 := \inf \left\{ \frac{\int_{\Gamma\backslash X} \|\text{grad } f\|^2 \, d\,\text{vol}}{\int_{\Gamma\backslash X} |f|^2 \, d\,\text{vol}} \ : \ f \in C_c^\infty(\Gamma\backslash X), \ f \neq 0 \right\}. \tag{2.4}$$

We call a λ_0-harmonic function on $\Gamma\backslash X$ a base eigenfunction. In general, a λ-harmonic function need not be a joint eigenfunction for the ring $\mathcal{D}(X)$. However, a square-integrable λ_0-harmonic function turns out to be a *positive* joint eigenfunction, up to a constant multiple. The following is obtained in [10, Corollary 6.6, Theorem 6.5] using Sullivan's work [24] and [14].

Theorem 2.3. [10] *If a base eigenfunction ϕ_0 belongs to $L^2(\Gamma\backslash X)$, then there exists $\psi \in \mathfrak{a}^*$ and a (Γ, ψ)-conformal measure ν on $\mathcal{F}$ such that ϕ_0 is proportional to E_ν.*

Here the space $L^2(\Gamma\backslash X)$ consists of square-integrable functions with respect to the inner product $\langle f_1, f_2 \rangle = \int_{\Gamma\backslash X} f_1 f_2 \, d\,\text{vol}$.

3. Higher rank smearing theorem

Let G be a connected semisimple real algebraic group and $\Gamma < G$ be a discrete subgroup. We recall the definition of a generalized Bowen-Margulis-Sullivan measure, as was defined in [6, Section 3].

Fix a pair of linear forms $\psi_1, \psi_2 \in \mathfrak{a}^*$. Let ν_1 and ν_2 be respectively (Γ, ψ_1) and (Γ, ψ_2) conformal measures on $\mathcal{F}$. Using the homeomorphism (called the Hopf parametrization) $G/M \to \mathcal{F}^{(2)} \times \mathfrak{a}$ given by $gM \mapsto (g^+, g^-, b = \beta_{g^-}(e, g))$, define the following locally finite Borel measure $\tilde{\mathsf{m}}_{\nu_1,\nu_2}$ on G/M as follows: for $g = (g^+, g^-, b) \in \mathcal{F}^{(2)} \times \mathfrak{a}$,

$$d\tilde{\mathsf{m}}_{\nu_1,\nu_2}(g) = e^{\psi_1(\beta_{g^+}(o,go)) + \psi_2(\beta_{g^-}(o,go))} \, d\nu_1(g^+) d\nu_2(g^-) db, \tag{3.1}$$

where $db = d\ell(b)$ is the Lebesgue measure on $\mathfrak{a}$. The measure $\tilde{\mathsf{m}}_{\nu_1,\nu_2}$ is left Γ-invariant and right A-semi-invariant: for all $a \in A$,

$$a_* \tilde{\mathsf{m}}_{\nu_1,\nu_2} = e^{(-\psi_1 + \psi_2 \circ \mathrm{i})(\log a)} \, \tilde{\mathsf{m}}_{\nu_1,\nu_2}, \tag{3.2}$$

where i denotes the opposition involution[3] $\mathrm{i} : \mathfrak{a} \to \mathfrak{a}$ (cf. [6, Lemma 3.6]). The measure $\tilde{\mathsf{m}}_{\nu_1,\nu_2}$ gives rise to a left Γ-invariant and right M-invariant measure on G by integrating along the fibers of $G \to G/M$ with respect to the Haar measure on M. By abuse of notation, we will also denote this measure by $\tilde{\mathsf{m}}_{\nu_1,\nu_2}$. We denote by

$$\mathsf{m}_{\nu_1,\nu_2} \tag{3.3}$$

[3] It is defined by $\mathrm{i}(u) = -\operatorname{Ad}_{w_0}(u)$ where w_0 is a Weyl group element with $\operatorname{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$.

the measure on $\Gamma\backslash G$ induced by $\tilde{\mathsf{m}}_{\nu_1,\nu_2}$, and call it the generalized BMS-measure associated to the pair (ν_1, ν_2).

The following theorem was proved in [10], extending the smearing argument due to Sullivan and Thurston ([25, Proposition 5], [5, Proof of Theorem 4.1]) to the higher rank setting.

Theorem 3.1 (Smearing theorem)**.** [10, Theorem 7.5] *For any pair (ν_1, ν_2) of Γ-conformal measures on $\mathcal{F}$, there exists $c > 0$ such that*

$$|\mathsf{m}_{\nu_1,\nu_2}| \le c \int_{1\text{-neighborhood of } \operatorname{supp} \mathsf{m}_{\nu_1,\nu_2}} E_{\nu_1}(x) E_{\nu_2}(x)\, d\operatorname{vol}(x).$$

An immediate corollary is as follows:

Corollary 3.2. *Let ν be a Γ-conformal measure on $\mathcal{F}$. If $|\mathsf{m}_{\nu,\nu}| = \infty$, then*

$$E_\nu \notin L^2(\Gamma\backslash X).$$

4. Proof of Main theorem

As in Theorem 1.1, let G be a connected semisimple real algebraic group with no rank one factors and $\Gamma < G$ be a Zariski dense discrete torsion-free subgroup. We recall the following recent theorem:

Theorem 4.1 (Fraczyk-Lee, [11])**.** *Suppose that $\operatorname{Vol}(\Gamma\backslash X) = \infty$. Then for any pair (ν_1, ν_2) of (Γ, ψ) and $(\Gamma, \psi \circ \mathrm{i})$-conformal measures for some $\psi \in \mathfrak{a}^*$,*

$$\mathsf{m}_{\nu_1,\nu_2}(\Gamma\backslash G) = \infty.$$

Corollary 4.2. *If $\operatorname{Vol}(\Gamma\backslash X) = \infty$, then for any pair (ν_1, ν_2) of Γ-conformal measures, $\mathsf{m}_{\nu_1,\nu_2}(\Gamma\backslash G) = \infty$.*

Proof. For $k = 1, 2$, let ν_k be a (Γ, ψ_k)-conformal measure with $\psi_k \in \mathfrak{a}^*$. Suppose $|\mathsf{m}_{\nu_1,\nu_2}| < \infty$. Since $a_* \mathsf{m}_{\nu_1,\nu_2} = e^{\psi_1(\log a) - \psi_2(\mathrm{i} \log a)} \mathsf{m}_{\nu_1,\nu_2}$ for all $a \in A$ by (3.2), it follows that

$$|\mathsf{m}_{\nu_1,\nu_2}| = e^{\psi_1(\log a) - \psi_2(\mathrm{i} \log a)} |\mathsf{m}_{\nu_1,\nu_2}|.$$

Since $|\mathsf{m}_{\nu_1,\nu_2}| < \infty$, we must have

$$\psi_2 = \psi_1 \circ \mathrm{i}.$$

Therefore the claim follows from Theorem 4.1. $\square$

Proof of Theorem 1.1 Suppose that $\operatorname{Vol}(\Gamma\backslash X) = \infty$ and ϕ_0 is a base eigenfunction in $L^2(\Gamma\backslash X)$. By Proposition 2.3, we may assume that $\phi_0 = E_\nu$ for some Γ-conformal measure ν on $\mathcal{F}$. Now by Theorem 3.1 and Corollary 4.2,

$$\infty = |\mathsf{m}_{\nu,\nu}| \ll \|E_\nu\|_2^2.$$

This is a contradiction.

Indeed, using a more precise version of the main theorem of [11] in replacement of Theorem 4.1, we obtain the following without the hypothesis on no rank one factors.

Theorem 4.3. *Let G be a connected semisimple real algebraic group and $\Gamma < G$ be a Zariski dense discrete subgroup. If $\Gamma\backslash X$ admits a square-integrable base eigenfunction, then $G = G_1G_2$, Γ is commensurable with $\Gamma_1\Gamma_2$ where G_1 (resp. G_2) is a product of rank one (resp. higher rank) factors of G, $\Gamma_1 < G_1$ is a discrete subgroup and $\Gamma_2 < G_2$ is a lattice.*

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